

POINCARÉ POLYNOMIAL OF MODULI SPACES OF FRAMED SHEAVES ON (STACKY) HIRZEBRUCH SURFACES

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ABSTRACT. We perform a study of the moduli space of framed torsion-free sheaves on Hirzebruch surfaces by using localization techniques. We discuss some general properties of this moduli space by studying it in the framework of Huybrechts-Lehn theory of framed modules. We classify the fixed points under a toric action on the moduli space, and use this to compute the Poincaré polynomial of the latter. This will imply that the moduli spaces we are considering are irreducible. We also consider fractional first Chern classes, which means that we are extending our computation to a stacky deformation of a Hirzebruch surface. From the physical viewpoint, our results provide the partition function of $N = 4$ Vafa-Witten theory on total spaces of the bundles $\mathcal{O}_{\mathbb{P}^1}(-p)$, which is relevant in black hole entropy counting problems according to a conjecture due to Ooguri, Strominger and Vafa.

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1. INTRODUCTION

In this paper we study the moduli space of framed torsion-free sheaves on a Hirzebruch surface \mathbb{F}_p . We start by studying the geometry of this moduli space within the framework of Huybrechts-Lehn's theory of framed modules. We then consider a natural toric action on the moduli space, determining its fixed points and the weight decomposition of the tangent spaces at the fixed points. This will in turn allow us to compute the Morse indexes at the fixed points, and therefore the Poincaré polynomial of the moduli space. As a byproduct, we show that the moduli spaces we consider are irreducible.

A conjectural formula for instanton counting on the total spaces of the bundles $\mathcal{O}_{\mathbb{P}^1}(-p)$ (which we denote X_p in this paper) was given in [12] and [14] using string theory techniques and two-dimensional reduction, respectively. Hirzebruch surfaces \mathbb{F}_p are projective compactifications of the spaces X_p , and indeed the results of this paper provide a proof of the conjectural formulas of [12, 14] for integer values of the first Chern class. Actually our computations also make sense for fractional first Chern classes (to be more precise, for Chern classes $c_1(E) = \frac{m}{p}C$, where m is an integer, and C is the “exceptional curve” in \mathbb{F}_p). This may be interpreted as a computation related to torsion-free sheaves on a stacky compactification of X_p , as we shall show in section 4. This relates to the stacky compactifications of ALE spaces discussed in [22].

The structure of this paper is as follows. In Section 2 we provide some physical motivations. In Section 3 we give some details about the structure of the moduli spaces we want to study. In Section 4 we compute the Poincaré polynomial of these moduli spaces. In Section 5 we draw some conclusions and check our formula for the Poincaré polynomial in some particular cases.

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2. PHYSICAL MOTIVATIONS

In this short section we want to provide some physical motivations and perspectives for the mathematical results we give in this paper. The reader who is not interested in this may safely skip to the next section.

The Ooguri-Strominger-Vafa conjecture [27] relates the counting of microstates of supersymmetric black holes to that of bound states of Dp-branes wrapping cycles of the internal Calabi-Yau threefold in string theory compactifications. For the D4-D2-D0 system, with the Euclidean D4 branes wrapping a four cycle M of the Calabi-Yau threefold, this counting problem has a rigorous mathematical definition in terms of the generating functional of the Euler characteristics of instanton moduli spaces on M with first and second Chern characters fixed by to the number of D0-branes and D2-branes respectively. In physical terms, this corresponds to the partition function of the Vafa-Witten twisted $N=4$ supersymmetric theory on M [30].

In [2] the example of a local Calabi-Yau modeled as the total space of a rank two bundle over a Riemann surface Σ_g was studied in detail, providing some support for the conjecture. Very little is known about the moduli space of instantons on such spaces; indeed the authors of [2] proceeded by conjecturing a reduction to the base Σ_g . An argument clarifying this reduction was provided in [4] via path-integral localization. Actually, rigorous calculations of the instanton partition function are available in a limited number of examples, namely the case of $p = 1$, where the blowup formulas of [30, 31] apply, and $p = 2$. Indeed, X_2 is the A_1 ALE space [18, 20] (for details on instanton counting on A_{p-1} ALE spaces the reader may refer to [18, 11, 12]).

One relevant feature of instantons on the spaces X_p is the appearance of solutions with fractional first Chern class. These are related to instantons that asymptote to flat connections with nontrivial holonomy at the boundary of X_p , which is topologically S^3/Γ . The flat connections are classified by irreducible representations of the finite group Γ . In this paper we propose an algebro-geometric counterpart of these solutions by considering the global quotient \mathbb{P}^2/Γ and performing a minimal resolution of the singularity at the origin. The resulting variety is a toric Deligne-Mumford stack \mathcal{X}_p whose coarse space is the Hirzebruch surface \mathbb{F}_p ; it corresponds to a “stacky” compactification of X_p obtained by adding a divisor $\tilde{C}_\infty \simeq \mathbb{P}^1/\Gamma$.

Torsion free sheaves framed on \tilde{C}_∞ provide solutions with fractional first Chern class. We will compute the Poincaré polynomial of the moduli space of these framed sheaves via localization and show that the resulting generating function of Euler characteristics is in agreement with the conjectural formulae of [12] and [14]. This has a nice expression in terms of modular forms. The details of this construction are presented in Sect.4.

3. MODULI OF FRAMED SHEAVES ON HIRZEBRUCH SURFACES

In this section we briefly study the structure of the moduli space of framed sheaves on Hirzebruch surfaces.

3.1. Hirzebruch surfaces. We denote by \mathbb{F}_p the p -th Hirzebruch surface $\mathbb{F}_p = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-p))$, which is the projective closure of the total space X_p of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-p)$ on \mathbb{P}^1 (a useful reference about Hirzebruch surfaces is [3]). This may be explicitly described as the divisor in $\mathbb{P}^2 \times \mathbb{P}^1$

$$\mathbb{F}_p = \{([z_0 : z_1 : z_2], [z : w] \in \mathbb{P}^2 \times \mathbb{P}^1 \mid z_1 w^p = z_2 z^p\},$$

Denoting by $f : \mathbb{F}_p \rightarrow \mathbb{P}^2$ the projection onto \mathbb{P}^2 , we let $C_\infty = f^{-1}(l_\infty)$, where l_∞ is the “line at infinity” $z_0 = 0$. The Picard group of \mathbb{F}_p is generated by C_∞ and the fibre F of the projection $\mathbb{F}_p \rightarrow \mathbb{P}^1$. One has

$$C_\infty^2 = p, \quad C_\infty \cdot F = 1, \quad F^2 = 0.$$

The canonical divisor K_p may be expressed as

$$K_p = -2C_\infty + (p - 2)F.$$

We shall consider the moduli space $\mathfrak{M}^p(r, k, n)$ parametrizing isomorphism classes of pairs (\mathcal{E}, ϕ) , where

- \mathcal{E} is a torsion-free coherent sheaf on \mathbb{F}_p , whose topological invariants are the rank r , the first Chern class $c_1(\mathcal{E}) = kC$, and the discriminant

$$\Delta(\mathcal{E}) = c_2(\mathcal{E}) - \frac{r-1}{2r} c_1^2(\mathcal{E}) = n;$$

- ϕ is a framing on C_∞ , i.e., an isomorphism of the restriction of \mathcal{E} to C_∞ with the trivial rank r sheaf on C_∞ :

$$\phi : \mathcal{E}|_{C_\infty} \xrightarrow{\sim} \mathcal{O}_{C_\infty}^{\oplus r}.$$

In constructing the moduli space $\mathfrak{M}^p(r, k, n)$ one only considers isomorphisms which preserve the framing, i.e., $(\mathcal{E}, \phi) \simeq (\mathcal{E}', \phi')$ if there is an isomorphism $\psi : \mathcal{E} \rightarrow \mathcal{E}'$ such that $\phi' \circ \psi = \phi$. While a point in $\mathfrak{M}^p(r, k, n)$ should always be denoted as the class of a pair (\mathcal{E}, ϕ) , we shall be occasionally a little sloppy in our notation, omitting the framing ϕ .

For $r = 1$, the value of k is irrelevant, and every moduli space $\mathfrak{M}^p(1, k, n)$ turns out to be isomorphic to the Hilbert scheme $X_p^{[n]}$ parametrizing length n 0-dimensional subschemes of X_p . The structure of the moduli space for $r > 1$ can be studied using an ADHM description

[28], which shows that the moduli space is a smooth algebraic variety of dimension $2rn$. However we are not going to give this description here.

3.2. Structure of the moduli space. A general result about the structure of moduli spaces of framed sheaves on a projective surface X was given by Nevins [26], who showed that, under some mild rigidity condition, there is a fine moduli scheme. However in the specific case at hand much more can be said. The structure of the moduli space may be studied by showing that, under some conditions, and after a suitable choice a polarization in X , and a certain stability parameter, framed sheaves yield stable pairs according to the theory developed by Huybrechts and Lehn [15, 16]. The moduli space turns out to be a quasi-projective separated scheme of finite type [7]. The obstruction to smoothness at a point $[(\mathcal{E}, \phi)]$ of the moduli space lies in the kernel of the trace map

$$\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}(-D)) \rightarrow H^2(X, \mathcal{O}(-D))$$

where D is the framing divisor. In our case $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}(-C_\infty)) = 0$, hence the moduli space is smooth (and then is an algebraic variety).

The result one gets in our case is the following.

Proposition 3.1. *The moduli space $\mathfrak{M}^p(r, k, n)$, when nonempty, is a smooth quasi-projective variety of dimension $2rn$. Its tangent space at a point $[\mathcal{E}]$ is isomorphic to the vector space $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}(-C_\infty))$.*

3.3. Toric action. The two-dimensional algebraic torus $\mathbb{C}^* \times \mathbb{C}^*$ acts on \mathbb{F}_p according to

$$([z_0 : z_1 : z_2], [z : w]) \xrightarrow{G_{t_1, t_2}} ([z_0 : t_1^p z_1 : t_2^p z_2], [t_1 z : t_2 w])$$

The divisor $C = f^{-1}([1 : 0 : 0])$ and C_∞ are fixed under this action. Note that one has $C = C_\infty - pF$ as divisors modulo linear equivalence. Moreover, this action has four fixed points, i.e., $p_1 = ([1 : 0 : 0], [0 : 1])$ and $p_2 = ([1 : 0 : 0], [1 : 0])$ lying on the exceptional line C , and two points lying on the line at infinity C_∞ . The invariance of C_∞ implies that the pullback G_{t_1, t_2}^* defines an action on $\mathfrak{M}^p(r, k, n)$. Moreover we have an action of the diagonal maximal torus of $Gl(r, \mathbb{C})$ on the framing. Altogether we have an action of the torus $T = (\mathbb{C}^*)^{r+2}$ on $\mathfrak{M}^p(r, k, n)$ given by

$$(t_1, t_2, e_1, \dots, e_r) \cdot (\mathcal{E}, \phi) = ((G_{t_1, t_2}^{-1})^* \mathcal{E}, \phi'), \quad (1)$$

where ϕ' is defined as the composition

$$(G_{t_1, t_2}^{-1})^* \mathcal{E}|_{C_\infty} \xrightarrow{(G_{t_1, t_2}^{-1})^* \phi} (G_{t_1, t_2}^{-1})^* \mathcal{O}_{C_\infty}^{\oplus r} \longrightarrow \mathcal{O}_{C_\infty}^{\oplus r} \xrightarrow{e_1, \dots, e_r} \mathcal{O}_{C_\infty}^{\oplus r}.$$

We study now the fixed point sets for the action of T on $\mathfrak{M}^p(r, k, n)$. This is basically the same statement as in [24] (see also [23] and [13]), but for the sake of completeness we give here a sketch of the proof.

Proposition 3.2. *The fixed points of the action (1) of T on $\mathfrak{M}^p(r, k, n)$ are sheaves of the type*

$$\mathcal{E} = \bigoplus_{\alpha=1}^r \mathcal{I}_\alpha(k_\alpha C) \quad (2)$$

where \mathcal{I}_α is the ideal sheaf of a 0-cycle Z_α supported on $\{p_1\} \cup \{p_2\}$ and k_1, \dots, k_r are integers which sum up to k . The α -th factor in this decomposition corresponds, via the framing, to the α -th factor in $\mathcal{O}_{C_\infty}^{\oplus r}$. Moreover,

$$n = \ell + \frac{p}{2r} \left(r \sum_{\alpha=1}^r k_\alpha^2 - k^2 \right) = \ell + \frac{p}{2r} \sum_{\alpha < \beta} (k_\alpha - k_\beta)^2 \quad (3)$$

where ℓ is the length of the singularity set of \mathcal{E} .

Proof. We first check that if \mathcal{E} is fixed under the T -action, then $\mathcal{E} = \bigoplus_{\alpha=1}^r \mathcal{E}_\alpha$, where each \mathcal{E}_α is a T -invariant rank-one torsion-free sheaf on \mathbb{F}_p . Let \mathcal{K} be the sheaf of rational functions on \mathbb{F}_p . Then $\mathcal{E}' = \mathcal{E} \otimes \mathcal{K}$ is free as a \mathcal{K} -module. Choose a trivialization $\mathcal{E}' \simeq \bigoplus_{\alpha=1}^r \mathcal{E}'_\alpha$ which, when restricted to C_∞ , provides an eigenspace decomposition for the action of T . Then set $\mathcal{E}_\alpha = \mathcal{E}'_\alpha \cap \mathcal{E}$, i.e., pick up the holomorphic sections of \mathcal{E}'_α . This provides the desired decomposition.

By taking the double dual of \mathcal{E} , and using T -invariance, we obtain $\mathcal{E}^{**} \simeq \bigoplus_{\alpha=1}^r \mathcal{O}_{\mathbb{F}_p}(k_\alpha C)$, whence \mathcal{E} has the form (2). Since the 0-cycles Z_α have to be T -invariant and should not be supported on C_∞ , they must be supported as claimed. The numerical equality (3) follows from a straight calculation. \square

The exact identification of the fixed points is obtained by using some Young tableaux combinatorics. The precursor of this technique is Nakajima's seminal book [21] where the case of the Hilbert scheme of points of \mathbb{C}^2 is treated (from a gauge-theoretic viewpoint, this is the “rank 1 case” for framed instantons on S^4). This was generalized to higher rank in [25, 10, 6].

A word on notation: if Y is a Young tableau, $|Y|$ will denote the number of boxes in it. In the case at hand, it turns out that to each fixed point one should attach an r -ple $\{Y_\alpha^{(i)}\}$ of pairs of Young tableaux (so $i = 1, 2$ and $\alpha = 1, \dots, r$). Write $Z_\alpha = Z_\alpha^{(1)} \cup Z_\alpha^{(2)}$, where $Z_\alpha^{(i)}$ is supported at p_i . The Young tableau $\{Y_\alpha^{(i)}\}$, for i and α fixed, is attached

to the ideal sheaf $\mathcal{I}_{Z_\alpha^{(i)}}$ as usual: choose local affine coordinates (x, y) around p_i and make a correspondence between the boxes of $\{Y_\alpha^{(i)}\}$ and monomials in x, y in the usual way (cf. [21]); then $\mathcal{I}_{Z_\alpha^{(i)}}$ is generated by the monomials lying outside the tableau. Now the identity (3) may be written as

$$n = \sum_{\alpha} (|Y_\alpha^1| + |Y_\alpha^2|) + \frac{p}{2r} \sum_{\alpha < \beta} (k_\alpha - k_\beta)^2. \quad (4)$$

Looking for all collections of Young tableaux and strings of integers k_1, \dots, k_r satisfying this condition together with $\sum_{\alpha=1}^r k_\alpha = k$, one enumerates all the fixed points.

3.4. Nonemptiness of the moduli space. The moduli spaces $\mathfrak{M}^p(r, k, n)$ may be given an ADHM description, as shown in [28]. The analysis performed in [28] implies the following results. Note that the value of k may be normalized in the range $0 \leq k < r - 1$ upon twisting by $\mathcal{O}_{\mathbb{F}_p}(C)$, and in the sequel we shall assume that this has been done.

Proposition 3.3. (i) *The moduli space $\mathfrak{M}^p(r, k, n)$ is nonempty if and only if the number $n - \frac{r-1}{2r}pk^2$ is an integer, and the bound*

$$n \geq N = \frac{pk}{2r}(r - k) \quad (5)$$

holds.

- (ii) *All sheaves in $\mathfrak{M}^p(r, k, N)$ are locally free.*
- (iii) *The moduli space $\mathfrak{M}^1(r, k, N)$ is isomorphic to the Grassmannian variety $G_k(r)$ of k -planes in \mathbb{C}^r . For $p > 1$, the space $\mathfrak{M}^p(r, k, N)$ is isomorphic to a rank $(p - 1)$ vector bundle on $G_k(r)$. In particular, for all $p \geq 1$ the moduli space $\mathfrak{M}^p(r, k, N)$ has the homotopy type of a compact space.*

4. POINCARÉ POLYNOMIAL

In this section we determine the weight decomposition of the toric action on the tangent space to the moduli space at the fixed points, and use this information to compute the Poincaré polynomial of the moduli spaces $\mathfrak{M}^p(r, k, n)$.

4.1. Defining the Poincaré polynomial. Let X be a space whose cohomology with rational coefficients is finite-dimensional. One defines the Poincaré polynomial of X as

$$P_t(X) = \sum_{n \geq 0} (\dim H^n(X, \mathbb{Q})) t^n.$$

4.2. Sheaves on stacky Hirzebruch surfaces. Actually our computations also make sense for $c_1(\mathcal{E}) = kC$ with $k = m/p$ for integer m , and $p \geq 2$. This can be justified by considering a “stacky compactification” of X_p ; instead of adding the divisor C_∞ , we add $\tilde{C}_\infty \simeq C_\infty/\mathbb{Z}_p$. One obtains a Deligne-Mumford stack \mathcal{X}_p , whose so-called coarse space may be identified with the Hirzebruch surface \mathbb{F}_p , and one has a morphism $\pi: \mathcal{X}_p \rightarrow \mathbb{F}_p$. Since the push-forward functor $\pi_*: \text{Coh}(\mathcal{X}_p) \rightarrow \text{Coh}(\mathbb{F}_p)$ is exact [1], one has an isomorphism

$$H^i(\mathcal{X}_p, \mathcal{F}) \simeq H^i(\mathbb{F}_p, \pi_*\mathcal{F}) \quad (6)$$

for any coherent sheaf $\mathcal{F} \in \text{Coh}(\mathcal{X}_p)$ and all i . Owing to this basic fact, we can reduce all our computations to the coarse space \mathbb{F}_p , by appropriately taking into account the push-forward of the relevant sheaves. The theory developed in [9] allows one to prove the following Lemma.

Lemma 4.1. *The Picard lattice of \mathcal{X}_p is freely generated over \mathbb{Z} by the divisor \tilde{C}_∞ and the inverse image $\tilde{F} = \pi^{-1}(F)$. Moreover one has*

$$\begin{aligned} \pi(p\tilde{C}_\infty) &= C_\infty \\ \pi(\tilde{F}) &= F. \end{aligned} \quad (7)$$

Let $\widetilde{\mathfrak{M}}^p(r, k, n)$ be the moduli space of torsion-free rank r sheaves \mathcal{E} on \mathcal{X}_p , with $c_1(\tilde{\mathcal{E}}) = kC$ and discriminant n , that are framed on \tilde{C}_∞ to the sheaf $\mathcal{O}_{\tilde{C}_\infty}^{\oplus r}$. Here $k = m/p$ for some integer m . The fixed points under the torus action are as in Proposition 3.2, except that in this case the k_α ’s are $k_\alpha = m_\alpha/p$, $m_\alpha \in \mathbb{Z}$.

Remark 4.2. We do not have for the moduli spaces $\widetilde{\mathfrak{M}}^p(r, k, n)$ a fully developed theory as we described in the previous section for the spaces $\mathfrak{M}^p(r, k, n)$. On the other hand, when k takes integer values, in the following treatment $\widetilde{\mathfrak{M}}^p(r, k, n)$ may be replaced by $\mathfrak{M}^p(r, k, n)$, so that all results are rigorous. Therefore, the results for non-integer values of k are to be regarded as heuristic, in the sense that they need a more rigorous proof, involving the construction of moduli spaces of framed sheaves on the stacks \mathcal{X}_p . \triangle

4.3. Weight decomposition of the tangent spaces. We compute here the weights of the action of the torus T on the irreducible subspaces of the tangent spaces to $\widetilde{\mathfrak{M}}^p(r, k, n)$ at the fixed points (\mathcal{E}, ϕ) of the action. According to the decomposition (2), the tangent space $T_{(E, \phi)}\widetilde{\mathfrak{M}}^p(r, k, n) \simeq \text{Ext}^1(\mathcal{E}, \mathcal{E}(-\tilde{C}_\infty))$ splits as

$$\text{Ext}^1(\mathcal{E}, \mathcal{E}(-C_\infty)) = \bigoplus_{\alpha, \beta} \text{Ext}^1(\mathcal{I}_\alpha(k_\alpha C), \mathcal{I}_\beta(k_\beta C - \tilde{C}_\infty)).$$

The factor $\text{Ext}^1(\mathcal{I}_\alpha(k_\alpha C), \mathcal{I}_\beta(k_\beta C - \tilde{C}_\infty))$ has weight $e_\beta e_\alpha^{-1}$ under the maximal torus of $Gl(r, \mathbb{C})$. So we need only to describe the weight decomposition with respect to the remaining action of $T^2 = \mathbb{C}^* \times \mathbb{C}^*$.

From the exact sequence $0 \rightarrow \mathcal{I}_\alpha \rightarrow \mathcal{O} \rightarrow \mathcal{O}_{Z_\alpha} \rightarrow 0$ we have in K-theoretic terms

$$\begin{aligned} \text{Ext}^\bullet(\mathcal{I}_\alpha(k_\alpha C), \mathcal{I}_\beta(k_\beta C - \tilde{C}_\infty)) &= \text{Ext}^\bullet(\mathcal{O}(k_\alpha C), \mathcal{O}(k_\beta C - \tilde{C}_\infty)) \\ &\quad - \text{Ext}^\bullet(\mathcal{O}(k_\alpha C), \mathcal{O}_{Z_\beta}(k_\beta C - \tilde{C}_\infty)) - \text{Ext}^\bullet(\mathcal{O}_{Z_\alpha}(k_\alpha C), \mathcal{O}(k_\beta C - \tilde{C}_\infty)) \\ &\quad + \text{Ext}^\bullet(\mathcal{O}_{Z_\alpha}(k_\alpha C), \mathcal{O}_{Z_\beta}(k_\beta C - \tilde{C}_\infty)). \end{aligned} \quad (8)$$

We should note that

$$\begin{aligned} \text{Ext}^0(\mathcal{O}(k_\alpha C), \mathcal{O}(k_\beta C - \tilde{C}_\infty)) &\simeq H^0(\mathcal{X}_p, \mathcal{O}(-n_{\alpha\beta}C - \tilde{C}_\infty)) \simeq \\ &\simeq H^0(\mathbb{F}_p, \mathcal{O}(-([n_{\alpha\beta}] + 1)C_\infty + pn_{\alpha\beta}F)) = 0, \end{aligned}$$

where $n_{\alpha\beta} = k_\alpha - k_\beta$ and $[n_{\alpha\beta}]$ is its integer part. In order to show the above vanishing we used the relation $C = p(\tilde{C}_\infty - F)$, the isomorphism (6) and the Lemma 7. More precisely, for our purposes we only need to compute the push-forward of line bundles; this is done along the lines of [5]. Analogously

$$\text{Ext}^2(\mathcal{O}(k_\alpha C), \mathcal{O}(k_\beta C - \tilde{C}_\infty)) \simeq H^0(\mathbb{F}_p, \mathcal{O}([n_{\alpha\beta}] + 1)C_\infty - pn_{\alpha\beta}F) = 0.$$

since the divisors C_∞ and F intersect. We are thus left with the computation of the Ext^1 groups

$$\text{Ext}^1(\mathcal{O}(k_\alpha C), \mathcal{O}(k_\beta C - \tilde{C}_\infty)) = H^1(\mathbb{F}_p, \mathcal{O}(-n_{\alpha\beta}C - \tilde{C}_\infty))$$

We distinguish three cases according to the values of $[n_{\alpha\beta}]$. In the first case ($[n_{\alpha\beta}] = 0$) one again easily sees that $H^1(\mathcal{X}_p, \mathcal{O}(-n_{\alpha\beta}C - \tilde{C}_\infty)) = H^1(\mathbb{F}_p, \mathcal{O}(p\{n_{\alpha\beta}\}F - C_\infty)) = 0$, where $\{n_{\alpha\beta}\}$ is the fractional part, $\{n_{\alpha\beta}\} = n_{\alpha\beta} - [n_{\alpha\beta}]$.

In the second case ($[n_{\alpha\beta}] > 0$) we get

$$H^1(\mathcal{X}_p, \mathcal{O}(-n_{\alpha\beta}C - \tilde{C}_\infty)) \simeq \bigoplus_{d=0}^{[n_{\alpha\beta}]-1} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(pd + p\{n_{\alpha\beta}\})) \quad (9)$$

We prove this result by induction on $[n_{\alpha\beta}] > 0$ using the exact sequence

$$0 \rightarrow \mathcal{O}(-(n_{\alpha\beta} + 1)C - \tilde{C}_\infty) \rightarrow \mathcal{O}(-n_{\alpha\beta}C - \tilde{C}_\infty) \rightarrow \mathcal{O}_C(-n_{\alpha\beta}C) \rightarrow 0 \quad (10)$$

For $n_{\alpha\beta} = 1$ one readily obtains (9). On the other hand from (10) we get

$$\begin{aligned} 0 \rightarrow H^0(\mathbb{P}^1, \mathcal{O}(n_{\alpha\beta}p)) &\rightarrow H^1(\mathcal{X}_p, \mathcal{O}(-(n_{\alpha\beta} + 1)C - \tilde{C}_\infty)) \\ &\rightarrow H^1(\mathcal{X}_p, \mathcal{O}(-n_{\alpha\beta}C - \tilde{C}_\infty)) \rightarrow H^1(\mathbb{P}^1, \mathcal{O}(n_{\alpha\beta}p)) = 0 \end{aligned}$$

and by the inductive hypothesis

$$\begin{aligned} 0 \rightarrow H^0(\mathbb{P}^1, \mathcal{O}(n_{\alpha\beta}p)) &\rightarrow H^1(\mathcal{X}_p, \mathcal{O}(-(n_{\alpha\beta} + 1)C - \tilde{C}_\infty)) \\ &\rightarrow \bigoplus_{d=0}^{[n_{\alpha\beta}] - 1} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(pd + p\{n_{\alpha\beta}\})) \rightarrow 0 \end{aligned}$$

so that (9) is proved. Since $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(pd + p\{n_{\alpha\beta}\}))$ is the space of homogeneous polynomials of degree $pd + p\{n_{\alpha\beta}\}$ in two variables, it equals $\sum_{i=0}^{pd+p\{n_{\alpha\beta}\}} t_1^{-i} t_2^{-pd+p\{n_{\alpha\beta}\}+i}$ in the representation ring of T^2 . Finally, we get the weights of this summand of the tangent space as

$$L_{\alpha,\beta}(t_1, t_2) = e_\beta e_\alpha^{-1} \sum_{i,j \geq 0, i+j-pn_{\alpha\beta} \equiv 0 \pmod{p}, i+j \leq p(n_{\alpha\beta}-1)} t_1^{-i} t_2^{-j} \quad (11)$$

For the third case ($n_{\alpha\beta} < 0$) we have analogously

$$L_{\alpha,\beta}(t_1, t_2) = e_\beta e_\alpha^{-1} \sum_{i,j \geq 0, i+j+2+pn_{\alpha\beta} \equiv 0 \pmod{p}, i+j \leq -pn_{\alpha\beta}-2} t_1^{i+1} t_2^{j+1} \quad (12)$$

These terms contribute to the weight decomposition of $T_{(E,\phi)} \widetilde{\mathfrak{M}}_p(r, k, n)$ at the fixed points together with the remaining terms in (8), which are essentially the same as in \mathbb{P}^2 case modulo some rescalings of the arguments (in a sense, we look at X_p as the resolution of singularities of $\mathbb{C}^2/\mathbb{Z}_p$, and rescale the arguments to achieve \mathbb{Z}_p -equivariance). In this way we get

$$\begin{aligned} T_{(E,\phi)} \widetilde{\mathfrak{M}}^p(r, k, n) = \\ \sum_{\alpha,\beta=1}^r \left(L_{\alpha,\beta}(t_1, t_2) + t_1^{p(k_\beta - k_\alpha)} N_{\alpha,\beta}^{\vec{Y}_1}(t_1^p, t_2/t_1) + t_2^{p(k_\beta - k_\alpha)} N_{\alpha,\beta}^{\vec{Y}_2}(t_1/t_2, t_2^p) \right), \quad (13) \end{aligned}$$

where $L_{\alpha,\beta}(t_1, t_2)$ is given by (11) and (12), while

$$N_{\alpha,\beta}^{\vec{Y}}(t_1, t_2) = e_\beta e_\alpha^{-1} \times \left\{ \sum_{s \in Y_\alpha} \left(t_1^{-l_{Y_\beta}(s)} t_2^{1+a_{Y_\alpha}(s)} \right) + \sum_{s \in Y_\beta} \left(t_1^{1+l_{Y_\alpha}(s)} t_2^{-a_{Y_\beta}(s)} \right) \right\}, \quad (14)$$

a well known expression for the \mathbb{C}^2 case (first introduced in [10]). Here \vec{Y} denotes an r -ple of Young tableaux, while for a given box s in the tableau Y_α , the symbols a_{Y_α} and l_{Y_α} denote the “arm” and “leg” of s respectively, that is, the number of boxes above and on the right to s .

4.4. Computing the Poincaré polynomial. We shall closely follow the method presented in [23], Section 3.4. We choose a one-parameter subgroup $\lambda(t)$ of the $r + 2$ dimensional torus T

$$\lambda(t) = (t^{m_1}, t^{m_2}, t^{n_1}, \dots, t^{n_r}),$$

by specifying generic weights such that

$$m_1 = m_2 \gg n_1 > n_2 \cdots > n_r > 0.$$

We have now a different fixed locus; as far as the action on the surface \mathbb{F}_p is concerned, the entire “exceptional line” C is pointwise fixed under this action. Accordingly, the fixed points in $\mathfrak{M}(r, k, n)$ still have the form (2), with Z_α fixed under the action of the diagonal subgroup Δ of $\mathbb{C}^* \times \mathbb{C}^*$, which means that the points in the supports of the 0-cycles Z_α are arbitrary points in C . Each component of the fixed point set is parametrized by an r -ple of pairs $((k_1, Y_1), \dots, (k_r, Y_r))$ with

$$n = \sum_{\alpha=1}^r |Y_\alpha| + \frac{p}{2r} \sum_{\alpha < \beta} (k_\alpha - k_\beta)^2 \quad \text{and} \quad \sum_{\alpha=1}^r k_\alpha = k.$$

Remark 4.3. This action allows one to prove that the space $\mathfrak{M}^p(r, k, n)$ has the homotopy type of a compact space, thus generalizing point (iii) of Proposition 3.3. This is needed to fully justify our computation of the Poincaré polynomial of $\mathfrak{M}^p(r, k, n)$ which uses, albeit indirectly, Morse theory. We may define a set

$$\mathfrak{M}_0^p(r, k, n) = \coprod_{0 \leq \ell \leq n-N} \mathfrak{M}_{\text{lf}}^p(r, k, n - \ell) \times \text{Sym}^\ell X_p$$

(where $\mathfrak{M}_{\text{lf}}^p(r, k, n)$ is the open subscheme of $\mathfrak{M}^p(r, k, n)$ given by the locally free points) and a map $\gamma: \mathfrak{M}^p(r, k, n) \rightarrow \mathfrak{M}_0^p(r, k, n)$ by letting

$$\gamma((\mathcal{E}, \phi)) = ((\mathcal{E}^{**}, \phi), \text{supp}(\mathcal{E}^{**}/\mathcal{E})).$$

The space $\mathfrak{M}_0^p(r, k, n)$ may be given a schematic structure by looking at it as a Uhlenbeck-Donaldson partial compactification of $\mathfrak{M}_{\text{lf}}^p(r, k, n)$, and the morphism γ is then projective (hence proper) [8, 17]. There is a natural action of \mathbb{C}^* on $\mathfrak{M}_0^p(r, k, n)$, which is compatible with the action on $\mathfrak{M}^p(r, k, n)$. The “deepest” stratum in $\mathfrak{M}_0^p(r, k, n)$ is isomorphic to $\mathfrak{M}^p(r, k, N) \times \text{Sym}^{n-N} X_p$, and its intersection Z with the fixed locus of the \mathbb{C}^* action is isomorphic to a product of symmetric products of copies of the exceptional curve C (hence, is compact). An analysis analogous to the one developed in [21, 24] shows that $\gamma^{-1}(Z)$ has the same homotopy type of $\mathfrak{M}^p(r, k, n)$. \triangle

To link with the previous construction, we should notice that each fixed point $\bar{\mathcal{E}}$ of the $\Delta \times T^r$ action corresponding to an r -ple $((k_1, Y_1), \dots, (k_r, Y_r))$ determines a fixed point $((k_1, \emptyset, Y_1), \dots, (k_r, \emptyset, Y_r))$ of the T -action which lies in the same component of the fixed locus as $\bar{\mathcal{E}}$. Since the $\Delta \times T^r$ -module structure of the tangent space $T_{(\mathcal{E}, \phi)} \widetilde{\mathfrak{M}}^p(r, k, n)$ at a fixed point of $\Delta \times T^r$ does not change if we move the fixed point in its component, we may choose $((k_1, \emptyset, Y_1), \dots, (k_r, \emptyset, Y_r))$. In this case (13) reduces to

$$T_{(E, \phi)} \widetilde{\mathfrak{M}}_p(r, k, n) = \sum_{\alpha, \beta=1}^r \left(L_{\alpha, \beta}(t_1, t_1) + t_1^{p(k_\beta - k_\alpha)} N_{\alpha, \beta}^{\vec{Y}}(1, t_1^p) \right), \quad (15)$$

By (14) we have

$$N_{\alpha, \beta}^{\vec{Y}}(1, t_1^p) = e_\beta e_\alpha^{-1} \times \left(\sum_{s \in Y_\alpha} t_1^{p(1 + a_{Y_\alpha}(s))} + \sum_{s \in Y_\beta} t_1^{-p a_{Y_\beta}(s)} \right).$$

Our task now is to compute the index of the critical points, that is, the number of terms in (15) for which one of the following possibilities holds:

- (i) the weight of t_1 is negative,
- (ii) the weight of t_1 is zero and the weight of e_1 is negative,
- (iii) the weights of t_1, e_1 are zero and the weight of e_2 is negative,
- (iv) the weights of t_1, e_1, e_2 are zero and weight of e_3 is negative,

... ..

($r + 1$) the weights of t_1, e_1, \dots, e_{r-1} are zero and the weight of e_r is negative.

The contribution of the diagonal ($\alpha = \beta$) terms of (15) turns out to be

$$\sum_{\alpha=1}^r (|Y_\alpha| - l(Y_\alpha)).$$

The nondiagonal ($\alpha \neq \beta$) terms can be rewritten as

$$\begin{aligned} \sum_{\alpha < \beta} \left(L_{\alpha, \beta}(t_1, t_1) + L_{\beta, \alpha}(t_1, t_1) \right. \\ \left. + \sum_{s \in Y_\alpha} \left(\frac{e_\beta}{e_\alpha} t_1^{p(1 + a_{Y_\alpha}(s) + k_\beta - k_\alpha)} + \frac{e_\alpha}{e_\beta} t_1^{p(-a_{Y_\alpha}(s) - k_\beta + k_\alpha)} \right) \right. \\ \left. + \sum_{s \in Y_\beta} \left(\frac{e_\beta}{e_\alpha} t_1^{p(-a_{Y_\beta}(s) + k_\beta - k_\alpha)} + \frac{e_\alpha}{e_\beta} t_1^{p(1 + a_{Y_\beta}(s) + k_\alpha - k_\beta)} \right) \right), \quad (16) \end{aligned}$$

We compute now the L terms in these expressions. Due to the formulas (11), (12), if $k_\alpha - k_\beta \geq 0$ only the term $L_{\alpha,\beta}(t_1, t_1)$ contributes to the index. This contribution is easy to count:

$$\sum_{i,j \geq 0, i+j-pn_{\alpha\beta} \equiv 0 \pmod{p}, i+j \leq p(n_{\alpha\beta}-1)} 1 = \frac{1}{2} [n_{\alpha\beta}] (p[n_{\alpha\beta}] + 2 - p) + p[n_{\alpha\beta}] \{n_{\alpha\beta}\}.$$

If, instead, $k_\alpha - k_\beta < 0$ there is a contribution from the $L_{\beta,\alpha}(t_1, t_1)$ term which is equal to

$$\sum_{i,j \geq 0, i+j > 0, i+j+pn_{\beta\alpha} \equiv 0 \pmod{p}, i+j \leq p(n_{\beta\alpha}-1)} 1 = \frac{1}{2} [n_{\beta\alpha}] (p[n_{\beta\alpha}] + 2 - p) + p[n_{\beta\alpha}] \{n_{\beta\alpha}\} - \delta_{p\{n_{\beta\alpha}\}, 0}.$$

Summarizing, the contribution of the L terms to the index is

$$l'_{\alpha,\beta} = \begin{cases} \frac{1}{2} [n_{\alpha\beta}] (p[n_{\alpha\beta}] + 2 - p) + p[n_{\alpha\beta}] \{n_{\alpha\beta}\} & \text{if } n_{\alpha\beta} \geq 0, \\ \frac{1}{2} [n_{\beta\alpha}] (p[n_{\beta\alpha}] + 2 - p) + p[n_{\beta\alpha}] \{n_{\beta\alpha}\} - \delta_{p\{n_{\beta\alpha}\}, 0} & \text{otherwise.} \end{cases} \quad (17)$$

The only operation to be yet performed is the sum over the boxes of Y_α in (16). A careful analysis shows that the contribution is $|Y_\alpha| + |Y_\beta| - n'_{\alpha,\beta}$, where

$$n'_{\alpha,\beta} = \begin{cases} \# \text{ of columns of } Y_\alpha \text{ that are longer than } k_\alpha - k_\beta \text{ if } k_\alpha - k_\beta \geq 0, \\ \# \text{ of columns of } Y_\beta \text{ that are longer than } k_\beta - k_\alpha - 1 \text{ otherwise.} \end{cases} \quad (18)$$

With this information we may compute the desired Poincaré polynomial.

Theorem 4.4. *The Poincaré polynomial of $\widetilde{\mathfrak{M}}^p(r, k, n)$ is*

$$P_t(\widetilde{\mathfrak{M}}^p(r, k, n)) = \sum_{\substack{\text{fixed} \\ \text{points}}} \prod_{\alpha=1}^r t^{2(|Y_\alpha| - l(Y_\alpha))} \prod_{i=1}^{\infty} \frac{t^{2(m_i^{(\alpha)} + 1)} - 1}{t^2 - 1} \prod_{\alpha < \beta} t^{2(l'_{\alpha,\beta} + |Y_\alpha| + |Y_\beta| - n'_{\alpha,\beta})}.$$

□

Here $m_i^{(\alpha)}$ is the number of columns in Y_α that have length i .

Corollary 4.5. *The generating function for the Euler characteristics of the moduli spaces $\widetilde{\mathfrak{M}}^p(r, k, n)$ is*

$$\sum_{k,n} P_{-1}(\widetilde{\mathfrak{M}}^p(r, k, n)) q^{n + \frac{pk^2}{2r}} z^k = \left(\frac{\theta_3\left(\frac{v}{p} \middle| \frac{\tau}{p}\right)}{\hat{\eta}(\tau)^2} \right)^r$$

with $q = e^{2\pi i \tau}$ and $z = e^{2\pi i v}$.

Proof. It just follows by setting $t = -1$ in the above expression for $P_t(\widetilde{\mathfrak{M}}^p(r, k, n))$ and the standard formulas for the (quasi)-modular functions

$$\begin{aligned}\theta_3(v|\tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi i v n} \\ \hat{\eta}(\tau) &= \prod_{l=1}^{\infty} (1 - q^l).\end{aligned}$$

□

5. SOME COMPARISONS AND CONSEQUENCES

5.1. Irreducibility of the moduli space. First, we note that our computation of the Poincaré polynomial of the spaces $\mathfrak{M}^p(r, k, n)$ allows us to conclude that these spaces are irreducible.

Theorem 5.1. *For all $p \geq 1$, and all values of the parameters r, k, n , the moduli space $\mathfrak{M}^p(r, k, n)$ of framed sheaves on the Hirzebruch surface \mathbb{F}_p is irreducible.*

Proof. By letting $t = 0$ in the formula for the Poincaré polynomial we see that $\mathfrak{M}^p(r, k, n)$ is connected, and since it is smooth (Proposition 3.1), it is also irreducible. □

5.2. Hilbert schemes of points. For $r = 1$ Theorem 4.4 yields the Poincaré polynomial of the Hilbert schemes of points of the spaces X_p . The generating function for these is given by

$$\sum_{n=0}^{\infty} P_t(X_p^{[n]}) q^n = \prod_{j=1}^{\infty} \frac{1}{(1 - t^{2j-2} q^j)(1 - t^{2j} q^j)} \quad (19)$$

This is independent of p , according to what was noted in [21, Cor. 7.7]: the Betti numbers of the Hilbert scheme of the total space of a line bundle on a Riemann surface Σ do not depend on the line bundle. Indeed formula (19) is the formula of [21, Cor. 7.7] for $\Sigma = \mathbb{P}^1$.

5.3. The rank 2 case. One can examine $\mathfrak{M}^p(2, k, n)$ in some detail. In the case $p = 1$ our result coincides with Corollary 3.19 of [23], where it is also shown (identity (3.20)) that the generating function of the Poincaré polynomial can be further factorized to obtain an elegant product formula. Our computations also agree with some explicit characterizations of the spaces $\mathfrak{M}^1(2, k, n)$ given in [7].

5.4. The case $p = 2$. This is examined in [29]. Besides usual line bundles $\mathcal{O}(k_\alpha C)$ with integer k_α , the author (without much mathematical justification) considers there also line bundles with half-integer Chern class k_α (see discussion after eq. (3.35) of [29]). This is what we get in the “stacky” case for $p = 2$.

It is interesting that incorporating the terms corresponding to the half-integer k_α again leads to a product formula (see (2.62)-(2.64) of [29]). This strongly suggests that one should be able to write a general factorized expression for the the generating function of the Poincaré polynomial in the case of the stacky Hirzebruch surfaces for all values of p .

5.5. The case of minimal discriminant. We would also like to compare our computation with the result in Proposition 3.3, which implies that whenever n reaches the lower bound $N = \frac{pk}{2r}(r - k)$, the moduli space $\mathfrak{M}^p(r, k, N)$ is homotopic to the Grassmannian variety $G_k(r)$. Coherently with the fact that all sheaves in $\mathfrak{M}^p(r, k, N)$ are locally free, in computing the Poincaré polynomial for this case one only considers empty Young tableaux, and takes values k_α that obey the condition

$$\sum_{\alpha=1}^r k_\alpha = \sum_{\alpha=1}^r k_\alpha^2.$$

This implies that each k_α is either 0 or 1. Thus the fixed points in the case (5) are in a one-to-one correspondence with the sequences $\vec{k} = (k_0, k_1, \dots, k_r)$ with $k_\alpha \in \{0, 1\}$ and $\#1 = k$. Such sequences may be conveniently represented as Young tableaux (below denoted as $Y_{\vec{k}}$) embedded in a rectangle of size $(r - k) \times k$ as follows: starting from the left top of the rectangle successively draw a vertical (horizontal) line segment of unit size if you encounter 0 (1). According to Theorem 4.4 Poincaré polynomial for this case is

$$P_t = \sum_{\vec{k}} \prod_{\alpha < \beta} t^{2l'_{\alpha, \beta}} \quad (20)$$

According to eq. (17) all $l'_{\alpha, \beta} = 0$, besides the cases with $k_\alpha = 1$ and $k_\beta = 0$ for which $l'_{\alpha, \beta} = 1$. But the number of latter cases exactly matches with the number of hooks of the given Young tableau. Since the number of hooks is the same as the number of boxes, we obtain

$$P_t = \sum_{\vec{k}} t^{2|Y_{\vec{k}}|}. \quad (21)$$

Thus the Poincare polynomial is nothing but the generating function (with respect to the parameter $q = t^2$) of the partitions with number of parts (i.e.length) $\leq r - k$ and with

largest part $\leq k$. This generating function is known to coincide with the “ q -binomial coefficient” [19, p. 27]

$$\begin{bmatrix} r \\ k \end{bmatrix} = \frac{(1 - q^{r-k+1}) \cdots (1 - q^r)}{(1 - q) \cdots (1 - q^k)},$$

which shows that the Poincaré polynomial (21) coincides with the Poincaré polynomial of the Grassmannian $G_k(r)$.

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